Performance of distributed algorithms for QoS in wireless ad hoc networks: arbitrary networks under the primary interference model, and line networks under the protocol interference model

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Abstract

Consider a wireless network where each communication link has a minimum bandwidth quality-of-service requirement. Certain pairs of wireless links interfere with each other due to being in the same vicinity, and this interference is modeled by a conflict graph. Given the conflict graph and link bandwidth requirements, the objective is to determine, using only localized information, whether the demands of all the links can be satisfied. At one extreme, each node knows the demands of only its neighbors; at the other extreme, there exists an optimal, centralized scheduler that has global information. The present work interpolates between these two extremes by quantitatively characterizing the tradeoff between the degree of decentralization and the performance of the distributed algorithm. This open problem is resolved for the primary interference model, and the following general result is obtained: if each node knows the demands of all links in a ball of radius $d$ centered at the node, then there is a distributed algorithm whose performance is away from that of an optimal, centralized algorithm by a factor of at most $(2d + 3)/(2d + 2)$. It is shown that for line networks under the protocol interference model, the row constraints are a factor of at most 3 away from optimal. Both bounds are best possible.

Index terms — graph theory, wireless ad hoc networks, distributed algorithms, admission control, quality-of-service, row constraints, primary interference model, protocol interference model, line networks

Contents

1 Introduction 2
  1.1 Model and problem formulation .......................... 3
  1.2 Summary of results ..................................... 5

2 Related work 6

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1. Introduction

Real-time applications such as voice and video require that the transmitted information be received without much delay. Such applications are called inelastic. The best-effort model of the internet, which does not make guarantees on timely delivery, is not sufficient for inelastic applications. The present work considers inelastic applications whose quality-of-service (QoS) requirements are specified in terms of the minimum bandwidth required by communication links in the network. The links are wireless, and wireless links in the same vicinity contend for the shared wireless medium. The bandwidth requirements are for links between nodes which are within communication radius of each other; the flows in this work refer to single-hop flows. When an application makes a demand for a certain quality-of-service, the network needs to determine whether there are sufficient resources to admit the new flow without disrupting the service to any previously admitted flow.

Consider an ad hoc network where nodes in the same vicinity directly communicate with each other, without any centralized infrastructure. The network can be modeled by an undirected graph $G = (V, L)$, where $V$ is a set of nodes and $L$ is a set of wireless links. The interference is modeled by a conflict graph $G_c = (L, L')$ whose vertices are the wireless links, and two wireless links are adjacent vertices in the conflict graph if and only if they cannot be simultaneously active due to interference. Suppose that each link $\ell$ makes a demand for bandwidth $f(\ell)$ b/s and that the total available bandwidth of the shared wireless medium is $C$ b/s. In the admission control problem studied in the present paper, the objective is to determine whether the demands of all the links can be satisfied. The scheduling problem, which is to obtain a link schedule satisfying the demands, is not considered in the present work.

Conflict graphs, which were introduced in [19], can be constructed from constraints imposed by the network’s MAC (medium access control) protocol. For example, in IEEE 802.11 MAC protocol-based networks, if a node $i$ is communicating with node $j$, then all nodes which are neighbors of $i$ or $j$ must remain idle while this communication takes place. This implies that two wireless links correspond to adjacent vertices in the conflict whenever they are at most one hop away from each other in the network connectivity graph. Another example is the primary interference model. In this
model, two links which are incident to a common node cannot be simultaneously active. Equivalently, the conflict graph is the line graph of the network graph. Under the primary interference model, the general resource allocation problem of assigning colors to the vertices of an arbitrary conflict graph is reduced to the special case of coloring the edges of the network graph. This problem has been well-studied in the literature \[10\] \[17\] and this interference model arises in some practical contexts \[31\] \[25\] \[21\] \[22\].

At one extreme, a centralized and optimal solution to the admission control problem exists: if the topology of the entire network and its conflict graph is known to a particular node and the demands of all the wireless links are also known to this center node, then this node can ascertain whether there exists a feasible schedule satisfying all the demands. However, there is a cost associated with communicating information from distant nodes to a center node, and determining an optimal solution is usually computationally expensive. Hence, it is desired that the solution be as decentralized as possible and that the admission control algorithm be efficient.

At the other extreme, each node in the conflict graph knows only the demands of all other nodes (wireless links) interfering with it. Sufficient conditions for distributed flow admission control in this setting include the row constraints \[15\] \[18\] and the scaled clique constraints \[15\]. Because only localized information is used, these conditions are conservative in the sense that they overestimate the resources required to satisfy the given demands. Flows may be denied admission even though they are physically feasible, i.e. feasible as per a centralized scheduler. Consequently, network resources are potentially underutilized. Ideally, the distributed admission control algorithm has as high an acceptance rate for flows as that of an optimal, centralized mechanism.

The present paper interpolates between these two extremes and quantitatively characterizes the trade-off between the performance of the distributed algorithm and the extent of decentralization for arbitrary networks under the primary interference model. Also, the worst-case performance of the row constraints is shown to be a bounded factor away from optimal for line networks under the protocol interference model.

1.1. Model and problem formulation

Consider a wireless network represented by a simple, undirected graph \( G = (V, L) \), where \( V \) is a set of wireless transceivers, also called nodes, and \( L \) is a set of wireless links. The edge set \( L \) consists of pairs of nodes which are within communication radius of each other. Nodes in the same vicinity cannot be simultaneously active due to wireless interference, and this interference is modeled by a conflict graph, as follows. The vertex set of the conflict graph \( G_c = (L, L') \) is the set of communication links in the network. Two vertices \( \ell_i, \ell_j \in L \) are adjacent in \( G_c \) whenever they cannot be simultaneously active due to wireless interference. In the sequel, \( G = (V, L) \) is referred to as the network graph and \( G_c = (L, L') \) as the conflict graph.

The quality-of-service (QoS) requirement is specified in terms of a minimum bandwidth requirement for each link. More specifically, suppose each link \( \ell \in L \) makes a
demand to transmit information at a certain data rate $f(\ell)$ b/s. Suppose the maximum transmission rate of link $\ell$ is $C(\ell)$ b/s. Then, the demand for link $\ell$ can be satisfied if link $\ell$ can be active for a fraction $\tau(\ell) := f(\ell)/C(\ell)$ of every unit of time. It is assumed throughout that $\tau(\ell)$ is a rational number.

The scheduling and flow admission control problems are now formally stated. An independent set in a graph is a subset of vertices which are pairwise nonadjacent. Because nonadjacent vertices in the conflict graph $G_c$ represent pairs of links which do not interfere with each other, an independent set in $G_c$ corresponds to a set of links which can be simultaneously active. Let $\chi$ fractional chromatic number defined to be the optimal value of the integer linear program

\[
\sum_{I \in \mathcal{I}(G_c)} \chi(I) \leq \tau(\ell) \quad \text{for total duration } T, \quad \text{and a particular link } \ell \in L \text{ is active for total duration } \sum_{I \in \mathcal{I}(G_c) : \ell \in I} t(I). \text{ If the total duration of each link } \ell \in L \text{ is at least } \tau(\ell), \text{ then we say there exists a schedule of duration } T \text{ satisfying link demand vector } (\tau(\ell) : \ell \in L). \text{ An optimal schedule for } \tau \text{ is a schedule of minimum duration satisfying demand } \tau.
\]

Given a conflict graph $G_c = (L, L')$ and link demand vector $\tau = (\tau(\ell) : \ell \in L)$, where $\tau(\ell)$ represents the fraction of every unit of time link $\ell$ is required to be active, let $T^*(G_c, \tau)$ denote the minimum duration of a schedule satisfying demand $\tau$. The demand $\tau$ is said to be feasible within duration $T$ if there exists a schedule of duration at most $T$ satisfying demand $\tau$, i.e. if $T^*(G_c, \tau) \leq T$. The demand $\tau$ is said to be feasible if there exists a schedule of duration at most 1 satisfying demand $\tau$. In the distributed admission control problem studied in the present paper, the problem is to determine, given $(G_c, \tau)$, whether there exists a schedule of duration at most 1 satisfying demand $\tau$, i.e. whether $T^*(G_c, \tau) \leq 1$, using only localized information. The problem of obtaining a schedule which satisfies this demand is not studied in this work. The independent set polytope $P_I = P_I(G_c)$ is defined to be the convex hull of the characteristic vectors of the independent sets in $G_c$. Thus, $P_I(G_c)$ is the set of all link demand vectors which are feasible within one unit of time.

An equivalent formulation of the problem in terms of the fractional chromatic number of a weighted graph is now given. For an introduction to fractional graph theory, the reader is referred to [28] [11]. Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, \ldots, v_n\}$, and let $\{I_1, \ldots, I_K\}$ denote the set of all independent sets in $G$. Define the incidence matrix $B = [b_{ij}]$ of $G$ by $b_{ij} = 1$ if $v_i \in I_j$ and $b_{ij} = 0$ if $v_i \notin I_j$. Thus, $B$ is a 01-matrix of size $n \times K$. The chromatic number of $G$, denoted $\chi(G)$, is defined to be the optimal value of the integer linear program

\[
\min 1^T x \text{ subject to } Bx \geq 1, \quad x \in \{0, 1\}^K.
\]

Relaxing the condition that $x$ be integral gives the linear program

\[
\min 1^T x \text{ subject to } Bx \geq 1, \quad x \geq 0,
\]

whose optimal value is called the fractional chromatic number of $G$ and is denoted $\chi_f(G)$. If $\tau = (\tau(v) : v \in V)$ is a set of nonnegative weights on the vertex set $V$, then
the fractional chromatic number of the weighted graph \((G, \tau)\), denoted \(\chi_f(G, \tau)\), is defined to be the optimal value of the linear program

\[
\min 1^T x \text{ subject to } Bx \geq \tau, x \geq 0.
\]

Given a conflict graph \(G_c = (L, L')\) and link demand vector \(\tau = (\tau(\ell) : \ell \in L)\), it is seen that the minimum duration of a schedule satisfying demand \(\tau\) is equal to the fractional chromatic number \(\chi_f(G_c, \tau)\).

The worst-case performance of a sufficient condition for admission control is defined as follows. Recall that the independent set polytope \(P_I\) of the conflict graph is the set of all link demand vectors \(\tau\) which are feasible. A necessary and sufficient condition for \(\tau\) to be feasible is that \(\tau \in P_I\). In general, determining whether \(\tau \in P_I\) is computationally intensive and requires global information. Hence, one is interested in obtaining sufficient conditions for admission control that can be implemented efficiently and in a distributed manner. Let \(S\) be any sufficient condition for admission control (particular examples of sufficient conditions are given in Section 3 and Section 4). Let \(P_S\) denote the set of all link demand vectors \(\tau\) which satisfy condition \(S\). Then, \(P_S \subseteq P_I\). Because condition \(S\) uses only localized information, it is often sub-optimal - it is conservative in the sense that it overestimates the amount of resources required to satisfy a given demand \(\tau\). One can scale the resource requirements in the sufficient condition \(S\) to obtain a necessary condition. The worst-case performance of the sufficient condition \(S\) is defined to be the smallest \(\alpha\) such that \(P_I \subseteq \alpha P_S\). The sufficient condition \(S\) is said to be a factor of at most \(\alpha\) away from optimal.

1.2. Summary of results

The main contributions of this paper are as follows.

1. A new distributed algorithm for admission control. A distance-\(d\) distributed algorithm for admission control is given for arbitrary networks when the interference is modeled using primary interference. The sufficient condition given can be implemented in a distributed fashion in the sense that each node needs to know the demands of only those links which are at most \(d\) hops away. The parameter \(d\) is the degree of centralization, and one can choose this parameter to be any value between 1 and the diameter of the network graph.

2. Worst-case performance analysis of the distance-\(d\) distributed algorithm. The distance-\(d\) distributed algorithm is shown to be a factor of at most \((2d+3)/(2d+2)\) away from an optimal, centralized algorithm. This extends previous results in [29] which focuses on the \(d = 0\) case, and results in [6] [7] where the \(d = 1\) case was studied. This result also resolves the open problem posed in [7, p. 1333], which asks to quantitatively characterize the tradeoff between the degree of localization of the distributed algorithm and the performance of the distributed algorithm.

3. Performance of row constraints in line networks under protocol interference model. It is shown that the row constraints are a factor of at most 3 away
from optimal for line networks under the protocol interference model. In [24]
[23], it was claimed that the conflict graphs arising in this context are claw-
free. However, this seems to be incorrect - in the present paper, an example is
given to show that there exist line networks for which, under the protocol in-
terference model, the conflict graph contains a claw. Consequently, polynomial
time scheduling algorithms that exist for claw-free graphs [26] [27] [4] are not
applicable. In the present paper, it is also shown that the bound of 3 is best
possible.

2. Related work

The design of distributed admission control mechanisms and distributed scheduling
mechanisms are well-known open problems and are an active area of research; see
[20] and references therein. Some distributed algorithms for admission control are
the row constraints [15] [18], the scaled clique constraints [15], and the degree and
mixed conditions [18]; the worst-case performance of these sufficient conditions was
analyzed in [7].

In the worst case, the row constraints are a factor of $\sigma(G_C)$ away from optimal,
where $\sigma(G_C)$ is defined below and is called the induced star number of the conflict
graph or the interference degree [9] [2]. In some cases, for example for conflict graphs
constructed from the 2-hop interference model of the IEEE 802.11 MAC protocol or
the K-hop interference model, the induced star number $\sigma(G_C)$ can be arbitrarily large
[2]. For some classes of networks and interference models, the induced star number is
bounded from above by a fixed constant [7] [2]; for example, the induced star number
of a unit disk graph is at most 5 [7] and the induced star number of a line graph is
at most 2.

In the special case where the wireless interference is modeled as primary inter-
ference, Shannon’s upper bound on the chromatic index of multigraphs [29] gives a
distributed admission control algorithm which is a factor of 1.5 away from optimal.
A distributed admission control mechanism was obtained in [7] and shown to be a
factor of 1.25 away from optimal.

A line network is one where all nodes are positioned in a straight line, say on the
$x$-axis. If each node has a certain radius of coverage and two nodes interfere with
each other whenever their coverage areas intersect, then one obtains a conflict graph
called an interval graph [1]. Conflict graphs arising in spectrum allocation are also
interval graphs [30]. Polynomial-time algorithms are known for finding maximum
weight independent sets in interval graphs.

Under the protocol interference model, recent work by Kose et al [24] [23] inves-
tigates computationally efficient solutions for the scheduling problem. The problem
of finding maximum weight independent sets in claw-free graphs is solvable in poly-
nomial time. The approach taken in [23] to address the situation where the conflict
graphs are not claw-free is to add edges to the conflict graph to obtain a claw-free
graph. This preserves the original interference constraints and gives a valid schedule
in polynomial time. Because of the added interference constraints, the throughput is
3. A distributed algorithm for the primary interference model

In this section, a new distributed algorithm for admission control is given and its performance is analyzed. The focus is on the primary interference model, and it is assumed that each node needs to know the quality-of-service requirements of only neighbors which are at distance at most $d$. In Section 3.1, the special cases of $d = 0$ and $d = 1$ are studied. A distance-$d$ distributed algorithm for admission control is given in Section 3.2 and its performance is analyzed in Section 3.3.

3.1. Preliminaries

One model of interference is the primary interference model. In this model, two links $\ell_i, \ell_j \in L$ in the network graph $G = (V, L)$ cannot be simultaneously active if and only if $\ell_i$ and $\ell_j$ have an endvertex in common. Equivalently, the conflict graph $G_c$ is the line graph of $G$. The line graph of a graph $G = (V, L)$, denoted $L(G)$, is the graph with vertex set $L$, and $\ell_i, \ell_j \in L$ are adjacent vertices in the line graph if and only if edges $\ell_i, \ell_j$ are incident to a common vertex in $G$.

Under the primary interference model, an independent set of vertices in the conflict graph $G_c = L(G)$ corresponds to a set of edges in the network graph which forms a matching. The minimum duration of a schedule satisfying link demand vector $\tau = (\tau(\ell) : \ell \in L)$ is called the fractional chromatic index of the edge-weighted graph $(G, \tau)$ and is denoted $T^* (\tau)$.

Example 1. Suppose the network graph is the odd cycle $C_n = (V, L)$, for some $n \geq 5$, where $L = \{\ell_0, \ell_1, \ldots, \ell_{n-1}\}$. Suppose the link demand vector is $\tau = (1, 1, \ldots, 1)$. Under the primary interference model, the maximum number of links which can be simultaneously active is $(n - 1)/2$. Hence, a lower bound for the minimum duration of a schedule satisfying demand $\tau$ is given by $T^* (C_n, \tau) \geq \frac{\sum_{\ell \in L} \tau(\ell)}{(n-1)/2} = 2n/(n - 1)$. A schedule whose duration is $2n/(n - 1)$ is now constructed. Let $\ell_0, \ell_1, \ldots, \ell_{n-1}$ be the links of the odd cycle in order. Under primary interference, the set $\{\ell_0, \ell_2, \ell_4, \ldots, \ell_{n-3}\}$ of $(n - 1)/2$ links can be simultaneously active. More generally, $A_i := \{\ell_i, \ell_{i+2}, \ell_{i+4}, \ldots, \ell_{i+n-3}\}$ is a maximal set of links which can be simultaneously active; here, subscripts are taken modulo $n$. The schedule $t$ defined by $t(A_i) = \frac{2}{n-1}$, $i = 0, 1, \ldots, n - 1$ has total duration $2n/(n - 1)$. Each link is in $(n - 1)/2$ of the $A_i$’s. Hence, each link is active for total duration $\frac{(n-1)}{2} \cdot \frac{2}{n-1} = 1$, as desired.

Given a network graph $G = (V, L)$ and a link demand vector $\tau = (\tau(\ell) : \ell \in L)$, define the degree of $\tau$ at node $v$ by $\delta(\tau, v) := \sum_{\ell : \ell \sim v} \tau(\ell)$, where the sum is over all links which are incident to node $v$. Define the maximum degree $\Delta(\tau) := \max_v \delta(\tau, v)$. Under primary interference, the time slots assigned to two links which are incident to
a common node must be disjoint. Thus, a lower bound for the duration of an optimal schedule is \( T^*(\tau) \geq \Delta(\tau) \). It follows that a necessary condition for demand \( \tau \) to be feasible is that the degree \( \delta(\tau, v) \) of \( \tau \) at every vertex \( v \) is at most 1.

A sufficient condition for feasibility is recalled next in Lemma 2. This result follows immediately from Shannon’s upper bound on the chromatic index of multigraphs \([29][5]\). A multigraph is a generalization of a graph obtained by allowing more than one (parallel) edge between pairs of vertices. The nontrivial part of Shannon’s proof is for the case where the maximum degree of the multigraph is odd. Because this part of the proof is not required to deduce Lemma 2 rather than apply Shannon’s result, an elementary proof from first principles is given below for Lemma 2. As shown in the proof, it can be assumed without loss of generality that the multigraphs arising in the admission control problem have even maximum degree.

**Lemma 2.** Let \( G = (V, L) \) be a network graph and let \( \tau = (\tau(\ell) : \ell \in L) \) be a link demand vector. Under the primary interference model, \( \tau \) is feasible within 1 unit of time if the degree of \( \tau \) at each vertex \( v \) is at most \( \frac{2}{3} \). This sufficient condition is a factor of at most 1.5 away from optimal.

**Proof:** Because the fraction of time \( \tau(\ell) \) a link \( \ell \) demands to be active is a rational number, one can divide the time axis into sufficiently small frames that the demand of each link is an integral number of frames. The time duration of each frame can be further halved so that the demand of a link \( \ell \) is an even integer \( \mu(\ell) \). Construct a multigraph \( M = (G, \mu) \) by replacing each edge \( \ell \) of \( G \) by \( \mu(\ell) \) parallel edges. Let \( \Delta(M) \) denote the maximum degree of the multigraph \( M \). Note that \( \Delta(M) \) is even. It can be assumed without loss of generality that \( M \) is \( \Delta(M) \)-regular, for one can add vertices and edges to obtain a \( \Delta(M) \)-regular multigraph. Suppose \( \Delta(M) = 2k \). By Petersen’s 2-factor theorem, the edge set of \( M \) can be decomposed into \( k \) 2-factors. Each 2-factor is a union of cycles and hence is 3-edge-colorable. Thus, the chromatic index of the multigraph \( M \) is at most \( 3k = 3\Delta(M)/2 \). An edge-coloring of the multigraph yields a valid schedule, and upper bounds on the chromatic index of the multigraph give corresponding upper bounds on the minimum duration of a schedule satisfying demand \( \tau \). Hence, \( T^*(\tau) \leq \frac{3}{2}\Delta(\tau) \). This proves that a sufficient condition for \( \tau \) to be feasible is \( \Delta(\tau) \leq \frac{2}{3} \).

Scaling the resource requirement of \( 2/3 \) in the sufficient condition \( \Delta(\tau) \leq \frac{2}{3} \) by a factor of 1.5 gives the condition \( \Delta(\tau) \leq 1 \), which is a necessary condition for \( \tau \) to be feasible. Hence, the sufficient condition in the assertion is a factor of at most 1.5 away from optimal.

The sufficient condition in Lemma 2 performs optimally, for example, when the network graph is a Shannon multigraph, essentially the thick triangle given in the next example.

**Example 3.** Suppose the network graph \( G = (V, L) \) is a 3-cycle graph and the link demand vector is \( \tau = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \). Then, \( \Delta(\tau) = \frac{2}{3} \), and so \( \tau \) is feasible by Lemma 2.

The proof of Lemma 2 decomposes the edge set of the network multigraph into 2-cycles and then uses the fact that each 2-cycle is 3-colorable. The resource requirement
of 3 colors is an overestimate if all the 2-cycles have even length. Therefore, in order to investigate the worst-case performance of the above sufficient condition, one can consider bipartite graphs. In fact, for bipartite graphs $G$, a necessary and sufficient condition for $\tau$ to be feasible is that $\Delta(\tau) \leq 1$. In the simplest case, consider a link demand vector of the form $\tau = (1, 0, \ldots, 0)$:

**Example 4.** Let $G$ be any network graph consisting of links $\ell_1, \ldots, \ell_m$ ($m \geq 1$). Consider the link demand vector $\tau = (1, 0, \ldots, 0)$. Because $\Delta(\tau) = 1 > \frac{2}{3}$, the distributed admission control algorithm of Lemma 2 will conclude that this demand cannot be satisfied. However, $\tau$ is clearly feasible.

Let $P_S = \{ \tau : \Delta(\tau) \leq \frac{2}{3} \}$. The smallest $\alpha$ for which $\alpha P_S := \{ \tau : \Delta(\tau) \leq \alpha \frac{2}{3} \}$ contains $(1, 0, \ldots, 0)$ is 1.5. Hence, in the worst case, the sufficient condition in Lemma 2 is a factor of at least 1.5 away from optimal. Also, $1.5P_S$ contains $P_I$. Hence, in the worst case, the sufficient condition in Lemma 2 is a factor of exactly 1.5 away from optimal.

A graph $G$ is said to be perfect if for each induced subgraph $H \subseteq G$, the chromatic number $\chi(H)$ and clique number $\omega(H)$ are equal. An odd hole in a graph is an induced odd cycle of length at least 5. An odd antihole in a graph is an induced subgraph which is the complement of an odd cycle of length at least 5. The strong perfect graph theorem asserts that a graph is perfect if and only if it does not contain any odd holes or odd antiholes. Given a conflict graph $G_c = (L, L')$ and demand $\tau = (\tau(\ell) : \ell \in L)$, a necessary condition for $\tau$ to be feasible is that $\tau(K) \leq 1$ for each clique $K$ in the conflict graph, where $\tau(K) := \sum_{\ell \in K} \tau(\ell)$. This necessary condition, called the clique constraints, can be scaled to give sufficient conditions for admission control [15], [7]. Let $P_{\text{clique}} = P_{\text{clique}}(G_C)$ denote $\{ \tau : \tau(K) \leq 1, \text{ for all } K \text{ in } G_C \}$. Then, $P_I = P_{\text{clique}}$ if and only if $G_C$ is a perfect graph [13]. Under primary interference, the following result gives a sufficient condition for admission control.

**Lemma 5.** [6] [7] [8] Let $G = (V, L)$ be a network graph and let $\tau = (\tau(\ell) : \ell \in L)$ be a link demand vector. Then, under the primary interference model, $\tau$ is feasible if the degree of $\tau$ at each vertex is at most 0.8 and the sum of the demands of links in every triangle in $G$ is at most 0.8. In the worst case, this sufficient condition is a factor of 1.25 away from optimal.

**Proof:** Under the primary interference model, the conflict graph $G_c$ is the line graph $L(G)$. The imperfection ratio of $G_c$, denoted $\text{imp}(G_c)$, is defined to be $\text{imp}(G_c) := \sup_{\tau \neq 0} \frac{\chi_f(G_c, \tau)}{\omega(G_c, \tau)}$, where $\omega(G_c, \tau)$ denotes the clique number of the vertex-weighted graph $(G_c, \tau)$, and the supremum is taken over all nonzero integral vectors $\tau$. If $G_c$ has no odd holes, then $\text{imp}(G_c) = 1$, and if the minimum length of an odd hole in $G_c$ is $g \geq 5$, then $\text{imp}(G_c) = \frac{g}{g-1}$ [10]. Because $g \geq 5$, one has $\text{imp}(G_c) \leq \frac{5}{4}$ and $\omega(G_c, \tau) \leq \chi_f(G_c, \tau) \leq 1.25 \omega(G_c, \tau)$. The following sufficient condition for admission control is thus obtained: $\tau$ is feasible if $\omega(G_c, \tau) \leq 0.8$. Each clique in the line graph $G_c = L(G)$ corresponds either to a set of links in $G$ which are incident to a common node in $G$ or to a set of links which forms a triangle in $G$. This gives the sufficient
condition in the assertion. It is clear this sufficient condition is a factor of at most 1.25 away from optimal.

Consider the 4-cycle network graph $G$ with link demand vector $\tau = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Because $\Delta(\tau) = 1 > 0.8$, the admission control protocol denies admission to $\tau$ even though $\tau$ is feasible. Let $P_S$ denote the set of demands for the said network graph which satisfy the sufficient condition in the assertion. The smallest value of $\alpha$ for which $\alpha P_S$ contains $\tau$ is 1.25. Hence, in the worst-case, the sufficient condition is a factor of at least 1.25 away from optimal. Scaling the sufficient condition by a factor of 1.25 gives a necessary condition.

Other network graphs $G$ for which the sufficient condition above exhibits its worst-case performance are those graphs $G$ for which the line graph $L(G)$ is perfect, because in such cases $\omega(G_c, \tau) = \chi_f(G_c, \tau)$. Trotter [32] showed that line graphs are perfect if and only if they do not contain odd holes. This implies that if the conflict graph does not contain any induced odd cycles of length at least 5, or equivalently, if the network graph does not contain odd cycles of length at least 5, then one obtains a sufficient condition for admission control which is also necessary by replacing the 0.8’s in Lemma 5 with 1’s. The factors 0.8 in the sufficient condition are due to the possibility of induced 5-cycles in $G_c$. Hence, the sufficient condition above performs optimally if one takes $G$ to be a 5-cycle graph with demand $\tau = (0.4, \ldots, 0.4)$.

The worst-case performance of the above sufficient condition is $\frac{5}{4}$. This bottleneck on performance is due to the possible existence of 5-cycles in the network graph which are not taken into consideration by the sufficient condition in Lemma 2 - only up to 3-cycles in the network graph are considered when estimating the resource requirements, and so a conservative factor of 0.8 is included to account for any 5-cycles which might exist in the network graph. One way to improve the performance of the distributed algorithm is to increase the amount of global information available at each node in the network graph. This is the approach taken in Section 3.2.

3.2. A distance-$d$ distributed algorithm

A distance-$d$ distributed algorithm is one which assumes that for each node $v \in V$ in the network graph $G = (V, L)$, node $v$ knows the demands of all communication links incident to it and the demands of all communication links $\ell = (x, y)$ between nodes $x$ and $y$ whenever the link’s endpoints $x$ and $y$ are at distance at most $d$ from $v$, and each node $v$ has no further global information. In this case, it is said that the degree of centralization is $d$. Thus, if the degree of centralization $d$ of a distributed algorithm is 0, then each node knows only the demands of all links incident to it; this special case was studied in Section 3.1 (see Lemma 2). If $d = 1$, then each node knows the demands of all links incident to it and the demands of all links between its neighbors (cf. Lemma 5).

A formal definition of the distance-$d$ distributed algorithm for admission control is as follows. Let $G = (V, L)$ be a network graph and let $v \in V$. Let $G_i(v)$ denote the set of vertices in $G$ whose distance to $v$ is exactly $i$. Thus, $G_0(v) = \{v\}$ and $G_1(v)$ is the set of neighbors of $v$. The subsets $G_0(v), G_1(v), \ldots$ are called the layers of the
distance partition of $G$ with respect to $v$. Define the ball of radius $d$ centered at $v$ by

$$W_{v,d} = \{v\} \cup G_1(v) \cup \cdots \cup G_d(v) \quad (d \geq 1).$$

Let $G_{v,d}$ denote the subgraph of $G$ induced by $W_{v,d}$. Let $\tau = (\tau(\ell) : \ell \in L)$ be a link demand vector. Let $T^*(\tau, G_{v,d})$ denote the minimum duration of a schedule satisfying the demands of all links in the induced subgraph $G_{v,d}$. Because such a schedule considers the demands of only links which are in the ball of radius $d$ centered at node $v$, its duration is a lower bound on $T^*(\tau)$. Note that $T^*(\tau, G_{v,d})$ does not depend on all of $\tau$ but only on the demands of those links which are in the distance-$d$ neighborhood $G_{v,d}$ of node $v$. Define $T^*_d(\tau) := \max_{v \in V} T^*(\tau, G_{v,d})$. Then, $T^*_d(\tau)$ is a lower bound for $T^*(\tau)$ and is a nondecreasing function of $d$.

The following is a distance-$d$ distributed algorithm for admission control: each node $v$ computes $T^*(\tau, G_{v,d})$ based on demands of all links which are in the ball of radius $d$ centered at $v$. If this quantity is less than some threshold (to be defined below), then the flows will be accepted; otherwise, the flows will be denied admission. The performance of this distributed algorithm is analyzed next.

### 3.3. Performance analysis

**Theorem 6.** Let $G = (V, L)$ be a network graph, let $\tau = (\tau(\ell) : \ell \in L)$ be a link demand vector, and let $d \geq 1$. Let $T^*(\tau, G_{v,d})$ denote the minimum duration of a schedule which satisfies the demands of all links in the distance-$d$ neighborhood graph $G_{v,d}$ of node $v$. Let $\alpha_d = \frac{2d+3}{2d+2}$. A sufficient condition for $\tau$ to be feasible is that $T^*(\tau, G_{v,d}) \leq \frac{1}{\alpha_d}$ for each node $v$. This sufficient condition is a factor of at most $\alpha_d$ away from optimal.

**Proof:** For the first part of the proof, it suffices to show that $\frac{T^*(\tau)}{T^*_d(\tau)} \leq \alpha_d$, where the numerator $T^*(\tau)$ is the fractional chromatic index of the edge-weighted graph $(G, \tau)$ and $T^*_d(\tau) = \max_{v \in V} T^*(\tau, G_{v,d})$; this would imply that any $\tau$ satisfying the sufficient condition has $T^*(\tau) \leq T^*_d(\tau) \alpha_d \leq \frac{1}{\alpha_d} \alpha_d = 1$. Let $W$ be a subset of $V$ of odd cardinality $k$, and let $E(G[W])$ denote the set of all links of $G$ in the induced subgraph $G[W]$. Define

$$\Lambda(\tau, W) := \frac{\sum_{\ell \in E(G[W])} \tau(\ell)}{(k-1)/2}.$$ 

Because the maximum size of a matching in the induced subgraph $G[W]$ is at most $(k-1)/2$, $\Lambda(\tau, W) \leq T^*(\tau)$. Define $\Lambda(\tau) := \max_{W \subseteq V} \Lambda(\tau, W)$, where the maximum is over all subsets $W \subseteq V$ of odd cardinality. It is clear that $\Lambda(\tau) \leq T^*(\tau)$. It follows from Edmonds’ Theorem (cf. [28] [3]) that $T^*(\tau) = \max\{\Delta(\tau), \Lambda(\tau)\}$.

It will now be proved that $\Delta(\tau) \leq \alpha_d T^*_d(\tau)$ and $\Lambda(\tau) \leq \alpha_d T^*_d(\tau)$; this would imply $T^*(\tau) \leq \alpha_d T^*_d(\tau)$. The minimum duration of a schedule satisfying the demands of all links in the distance-$d$ neighborhood $G_{v,d}$ of node $v$ is at least the degree $\delta(\tau, v)$ of $\tau$ at node $v$. Hence, $\Delta(\tau) \leq T^*_d(\tau)$, which is at most $\alpha_d T^*_d(\tau)$ because $\alpha_d \geq 1$. The inequality $\Lambda(\tau) \leq \alpha_d T^*_d(\tau)$ will be proved next. Fix $k$ such that $3 \leq k \leq |V(G)|$ and
Proof: Let \( k \) be a subset of \( V(G) \) of cardinality \( k \). It suffices to show that

\[
\frac{\sum_{\ell \in E(G[W])} \tau(\ell)}{(k-1)/2} \leq \alpha_d T_d^*(\tau).
\]

Consider two cases for \( k \) (this proof technique is from [10]). First suppose that \( 3 \leq k < 2d + 3 \). The maximum size of a matching in \( G[W] \) is at most \((k-1)/2\), whence the minimum duration of a schedule satisfying the link demand vector \( (\tau(\ell) : \ell \in E(G[W])) \) is at least \( \frac{1}{(k-1)/2} \sum_{\ell \in E(G[W])} \tau(\ell) \). Because \( k \leq 2d + 1 \), each connected component in the induced subgraph \( G[W] \) is contained in some ball of radius \( d \) in \( G \). Hence, the minimum duration of a schedule satisfying the link demand vector \( (\tau(\ell) : \ell \in E(G[W])) \) is at most \( T_d^*(\tau) \). It follows that

\[
\frac{\sum_{\ell \in E(G[W])} \tau(\ell)}{(k-1)/2} \leq T_d^*(\tau) \leq \alpha_d T_d^*(\tau).
\]

Now suppose \( k \geq 2d + 3 \), where \( |W| = k \) again. One can double-count the sum of demands of all links in \( E(G[W]) \) (this is essentially the handshaking theorem) to obtain that \( 2 \sum_{\ell \in E(G[W])} \tau(\ell) \leq k\Delta(\tau) \). Hence,

\[
\frac{\sum_{\ell \in E(G[W])} \tau(\ell)}{(k-1)/2} \leq \frac{(k/2)\Delta(\tau)}{(k-1)/2} \leq \frac{k}{k-1} T_d^*(\tau) \leq \alpha_d T_d^*(\tau),
\]

where the last two inequalities follow from the fact that \( \Delta(\tau) \leq T_d^*(\tau) \) and that \( k/(k-1) \) is a decreasing function of \( k \). This proves that \( \Lambda(\tau) \leq \alpha_d T_d^*(\tau) \), as required.

Thus, for any given network graph \( G = (V, E) \), link demand vector \( \tau \), and distance \( d \geq 1 \), one has \( T^*(\tau) \leq \alpha_d T_d^*(\tau) \). In particular, if \( T_d^*(\tau) \leq \frac{1}{\alpha_d} \), then \( T^*(\tau) \leq 1 \). This gives the sufficient condition for admission control in the assertion in which each node uses localized information up to distance \( d \) in the network graph. Scaling the right-hand side of the sufficient condition \( T^*(\tau, G_{v,d}) \leq \frac{1}{\alpha_d} \) by a factor of \( \alpha_d \) gives \( T^*(\tau, G_{v,d}) \leq 1 \), which is a necessary condition for \( \tau \) to be feasible. Hence, the sufficient condition in the assertion is a factor of at most \( \alpha_d \) away from optimal.

The bound \( \alpha_d \) on the performance of the above sufficient condition is best possible:

**Proposition 7.** In the worst case, the sufficient condition in Theorem [6] is a factor of exactly \( \alpha_d \) away from optimal.

**Proof:** Let \( G \) be any network graph and let \( d \geq 1 \). Let \( P_S = \{ \tau : T_d^*(\tau) \leq \frac{1}{\alpha_d} \} \) be the set of all link demand vectors which satisfy the sufficient condition in Theorem [6]. The link demand vector \( \tau = (1, 0, \ldots, 0) \) is feasible but is not contained in \( P_S \). The smallest value of \( \alpha \) for which \( \alpha P_S \) contains \( \tau \) is \( \alpha_d \). This proves that in the worst case, the sufficient condition in Theorem [6] is a factor of at least \( \alpha_d \) away from optimal. By Theorem [6], the sufficient condition is also a factor of at most \( \alpha_d \) away from optimal. \[\square\]
4. Performance of row constraints in line networks under the protocol interference model

A simple, distributed mechanism for both admission control and scheduling is given by the row constraints, which can be obtained by generalizing the greedy graph coloring algorithm to weighted graphs, as follows. Let $G_c = (L, L')$ be a conflict graph and let $\tau = (\tau(\ell) : \ell \in L)$ be a link demand vector. A sufficient condition for $\tau$ to be feasible within duration $T$ is that $\tau(\ell) + \tau(\Gamma(\ell)) \leq T$, for all $\ell \in L$, where $\Gamma(\ell)$ denotes the set of neighbors in $G_c$ of vertex $\ell$, and $\tau(\Gamma(\ell)) := \sum_{\ell' \in \Gamma(\ell)} \tau(\ell')$. This sufficient condition is called the row constraints (cf. \cite{15} \cite{18} \cite{33}). If $\tau$ is feasible, then a feasible schedule can be obtained by allocating any time interval to $\ell$ which is disjoint from the time intervals allocated to its neighbors in $G_c$.

While the problem of computing the exact value of the fractional chromatic number $T^*(G_c, \tau)$ is NP-hard \cite{12}, the quantity $\max_{\ell \in L} \{\tau(\ell) + \tau(\Gamma(\ell))\}$ can be computed efficiently and is an upper bound on $T^*(G_c, \tau)$. The worst-case performance of the row constraints is the largest possible factor by which the row constraints can overestimate the resources required for accepting demand $\tau$, and is equal to $\sup_{\tau \in \mathcal{P}_I} \max_{\ell \in V(G_c)} \{\tau(\ell) + \tau(\Gamma(\ell))\}$. A simple formula for this expression was obtained in \cite{2} and independently in \cite{8} \cite{6} \cite{9}. This result is recalled in Lemma \ref{lemma:rowconstraints} below, which says that in the worst case, the row constraints are a factor of $\sigma(G_c)$ away from optimal, where the graph invariant $\sigma(G_c)$, defined below, is called the induced star number of $G_c$. Thus, it is desired that this quantity be as small as possible.

In Section 4.1, the definition of the induced star number of a graph and a result on the worst-case performance of the row constraints are recalled. The protocol interference model is also described. In Section 4.2, it is shown that for line networks under the protocol interference model, the row constraints are a factor of at most 3 away from optimal. Further, it is shown that this bound is best possible.

4.1. Preliminaries

In this section, results on the induced star number of a graph and the protocol interference model are recalled. The induced star number of a graph $G$, denoted by $\sigma(G)$, is the number of leaf vertices in the maximum sized induced star subgraph of $G$, i.e.

$$\sigma(G) := \max_{v \in V} \alpha(G[G_1(v)])$$

where $\alpha(G)$ denotes the independence number of $G$, $G[W]$ denotes the subgraph of $G$ induced by $W \subseteq V$, and $G_1(v)$ denotes the set of neighbors in $G$ of vertex $v$. The induced star number of the conflict graph, denoted by $\sigma(G_c)$, is referred to in \cite{2} as the interference degree of the network and is the maximum number of pairwise noninterfering links in $L$ which interfere with a common link.
Lemma 8. Let $G_c = (L, L')$ be any graph. Then,

$$\sup_{\tau \neq 0} \frac{\max_{\ell \in V(G_c)} \{\tau(\ell) + \tau(\Gamma(\ell))\}}{T^*(G_c, \tau)} = \sigma(G_c).$$

Lemma 8 says that in the worst case, the row constraints are a factor of at most $\sigma(G_C)$ away from optimal.

For some classes of networks and interference models such as tree networks under the protocol interference model \[24, Theorem 3\] and Bluetooth and sensor networks \[18, Example 5\], the conflict graph $G_C$ is a disjoint union of complete graphs, i.e. $\sigma(G_C) = 1$. The graph $K_{1,3}$ is called a claw. A graph is claw-free if it does not contain a claw as an induced subgraph. Equivalently, $G_C$ is claw-free if and only if $\sigma(G_C) \leq 2$. The induced star number of a line graph is at most 2, and so every line graph is claw-free. Claw-free graphs were initially studied as a generalization of line graphs.

The protocol interference model is described next. Let $V$ be the set of nodes of a wireless network. Assume that each node has the same transmission range $r_T$. Let $N(i)$ denote the set of nodes within transmission range of node $i \in V$. A (multicast) transmission is defined to be a pair $(i, J)$ where $i \in V$ is called the transmitter and $J \subseteq V$ is a set of receivers. A transmission $(i, J)$ is said to be valid if $J \subseteq N(i)$, i.e. if each of the receivers is within the radius of coverage of the transmitter.

Given a set $\{(i_k, J_k) : k \in K\}$ of valid transmissions over some index set $K$, suppose some pairs of transmissions cannot be scheduled simultaneously due to wireless interference. One can construct a conflict graph $G_c$ to capture which transmissions interfere with each other. Each transmission corresponds to a vertex in the conflict graph $G_c$, and two vertices are adjacent in $G_c$ if and only if the corresponding two transmissions interfere with each other. Each transmission corresponds to a vertex in the conflict graph $G_c$, and two vertices are adjacent in the conflict graph $G_c$ if and only if the corresponding two transmissions interfere with each other. Assume the positions of the nodes are known, so that the distance $\text{dist}(i, j)$ between any pair of nodes $i, j \in V$ can be computed.

**Construction 9. Conflict graph of a network under the protocol interference model.** Given a set $V$ of wireless nodes in the Euclidean plane and a set $\{(i_k, J_k) : k \in K\}$ of valid transmissions, the conflict graph $G_c$ is constructed as follows. The vertex set of $G_c$ is the set of valid transmissions; denote the transmission $(i_k, J_k)$ by the vertex $v_k \in V(G_c)$. Two vertices $v_1 = (k_1, J_1)$ and $v_2 = (i_2, J_2)$ are defined to be adjacent in $G_c$ if and only if any one of the following conditions is satisfied:

(a) $i_1 = i_2$
(b) $i_1 \in J_2$ or $i_2 \in J_1$
(c) $J_1 \cap J_2 \neq \emptyset$
(d) $\text{dist}(i_2, j) < \text{dist}(i_1, j)$ for some $j \in J_1$
(e) $\text{dist}(i_1, j) < \text{dist}(i_2, j)$ for some $j \in J_2$.

Implicit in this construction of the conflict graph is the following model of interference, usually referred to as the protocol interference model (cf. [14]). Condition (a) says
that a node has at most one transmitter, and (b) says that a node cannot transmit and receive at the same time. Condition (c) models the constraint that a node has at most one receiver. In order for a transmission \((i_1, J_1)\) to be successful, each receiver in \(J_1\) must be closer to transmitter \(i_1\) than to any other transmitter. If any receiving node in \(J_1\) is closer to another transmitter \(i_2\), then the interference is considered to be intolerable and the transmissions \((i_1, J_1)\) and \((i_2, J_2)\) are said to interfere with each other. This conflict is captured by condition (d). Similarly, condition (e) captures the interference experienced by the receivers in \(J_2\).

In the next section, it is shown that \(\sigma(G_c) \leq 3\) for certain classes of networks. These are straight line networks for which the demands may be for unicast or multicast transmissions, and the protocol interference model defines which links interfere with each other. The conflict graph of line networks under the protocol interference model has been studied recently in Köse and Médard [24] and Köse et al. [23]. The proof of the result in [24] [23] that the conflict graphs arising in this context are claw-free seems to be incorrect; as proved in Theorem 11 below, there exists a line network for which the conflict graph contains a claw.

4.2. Performance analysis

In this section, the focus is on line networks, which are networks satisfying the condition that all the wireless nodes lie on the same line, say on the \(x\)-axis. Recall that the row constraints give a sufficient condition for distributed admission control. In this section, it is shown that given a line network, if the interference is modeled by the protocol interference model, then the row constraints are a factor of at most 3 away from optimal (cf. Theorem 15).

Denote the position of node \(A \in V\) by \(x_{pos}(A)\). The closed interval \(\{x : a \leq x \leq b\}\) on the real number line is denoted by \([a, b]\). In this section, it is assumed that sinks are placed to the right of the source, so that information travels towards the right (eastward) and hence each transmission \((i_k, J_k)\) satisfies the property that the \(x\)-coordinate of node \(i_k\) is at most the \(x\)-coordinate of each node in \(J_k\).

**Lemma 10.** Let \(v_1 = (A, B)\) be the transmission from node \(A\) to node \(B\), and let \(v_2 = (C, D)\) denote the transmission from node \(C\) to node \(D\). Without loss of generality, suppose the second transmitter \(C\) lies to the right of (or in the same position as) the first transmitter \(A\) on the real number line, i.e. \(x_{pos}(C) \geq x_{pos}(A)\). Then, in the conflict graph constructed from conditions (a)-(e) of Construction 9 above,

1. transmissions \(v_1\) and \(v_2\) are adjacent vertices if and only if \(x_{pos}(C) \in [x_{pos}(A), 2x_{pos}(B) - x_{pos}(A)]\).

2. transmissions \(v_1\) and \(v_2\) are nonadjacent in the conflict graph if and only if \(x_{pos}(C) > x_{pos}(B) + s\), where \(s\) is the distance between nodes \(A\) and \(B\).

**Proof:** Construction 9 is such that two transmissions \((A, B)\) and \((C, D)\) will be nonadjacent in the conflict graph only if receiver \(B\) is closer to transmitter \(A\) than to transmitter \(C\). This condition is violated precisely when either (i) the intervals \([x_{pos}(A), x_{pos}(B)]\) and intervals \([x_{pos}(C), x_{pos}(D)]\) overlap, or (ii) if these two intervals are disjoint and the distance from \(B\) to \(C\) is at most the distance \(s\) from \(B\) to \(A\);
Figure 1: Conditions under which receiver $B$ experiences interference due to transmitter $C$.

(i)

(ii)

Figure 2: A line network whose conflict graph contains a claw. The $x$-coordinates of nodes and distances between adjacent nodes are also shown.

see Figure 1. In case (i), there is interference at receiver $B$ because it is closer to transmitter $C$ than to transmitter $A$. Case (ii) occurs exactly when $x_{\text{pos}}(C) \leq x_{\text{pos}}(B) + s$. This proves (1), and (2) follows immediately.

**Theorem 11.** Let $r_T$ denote the transmission radius of each wireless node. Consider a line network with nodes $A_1, A_2, \ldots, A_n$ positioned on the $x$-axis in such a manner that each node $A_i$ is able to transmit to at most 2 nodes to its right, i.e. $x_{\text{pos}}(A_{i+3}) - x_{\text{pos}}(A_i) > r_T$ for all $i = 1, 2, \ldots, n - 3$. Let $G_c$ denote the conflict graph of a set of valid transmissions in this network. Then, there exists a line network for which the conflict graph $G_c$ contains a claw.

**Proof:** Consider the line network shown in Figure 2 consisting of 8 nodes $A_1, \ldots, A_8$, positioned at locations 0, 0.3, 0.5, 1.4, 1.5, 1.6, 2.49 and 2.51, respectively. Let the transmission radius of each node be $r_T = 1$. The nodes in these positions satisfy the condition that each node can communicate to at most two nodes to its right. It can be verified that the 4 transmissions $v_1 = (A_3, A_5)$, $v_2 = (A_1, A_2)$, $v_3 = (A_4, A_6)$ and $v_4 = (A_7, A_8)$ are valid (in the sense that each receiver is within communication radius of its transmitter), and that they form a claw in the conflict graph with center vertex $v_1$.

The problem of bounding the induced star number of the conflict graph becomes easier if it can be assumed that the set of valid transmissions consists of distinct
unicast transmissions. The next two results establish that this assumption can be made without loss of generality because the induced star number remains the same if multicast transmissions are replaced with corresponding unicast transmissions.

**Lemma 12.** Consider a line network having a valid multicast transmission \( v = (A, \{ B, C \}) \) and a valid unicast transmission \( w = (D, E) \), where the \( x \)-coordinate of node \( B \) is less than the \( x \)-coordinate of node \( C \); see Figure 3. Define a new unicast transmission \( v' = (A, C) \). Suppose Construction 9 is used to determine which pairs of transmissions interfere with each other. Then, unicast transmission \( w \) and multicast transmission \( v \) interfere with each other if and only if unicast transmission \( w \) and unicast transmission \( v' \) interfere with each other.

**Proof:** Consider the intervals \( \alpha = [x_{\text{pos}}(A), x_{\text{pos}}(C)] \) and \( \delta = [x_{\text{pos}}(D), x_{\text{pos}}(E)] \). As one slides \( \delta \) from left to right on the \( x \)-axis, a few cases arise.

First, suppose \( \delta \) is to the left of and disjoint from \( \alpha \). Then, transmissions \( w \) and \( v \) do not interfere with each other if and only if receiver \( E \) is closer to transmitter \( D \) than to transmitter \( A \), which is the case if and only if transmissions \( w \) and \( v' \) do not interfere with each other. Observe that the fact that node \( B \) is not a receiver in transmission \( v' \) does not play a role in determining whether the two transmissions \( w \) and \( v \) (or \( w \) and \( v' \)) interfere with each other because any interference between the two transmissions occurs at receivers \( C \) or \( E \) and is due to transmitters \( A \) or \( D \).

Next, suppose \( x_{\text{pos}}(E) = x_{\text{pos}}(A) \). Then by rule (b) of Construction 9 transmissions \( w \) and \( v \) interfere with each other, as do transmissions \( w \) and \( v' \).

If \( x_{\text{pos}}(E) \in \alpha \) and \( E \neq B \), then there exist two distinct nodes, namely \( E \) and \( B \), between transmitter \( A \) and receiver \( C \), contradicting the fact that a node can’t transmit to more than two nodes to its right.

Now suppose \( x_{\text{pos}}(E) \in \alpha \) and \( E = B \). Because node \( A \) can’t transmit to more than two nodes to its right, one has \( x_{\text{pos}}(D) \leq x_{\text{pos}}(A) \), and so receiver \( E \) will experience interference due to transmitter \( A \).

Finally, if \( x_{\text{pos}}(E) \geq x_{\text{pos}}(C) \), then any interference between transmissions \( w \) and \( v \) is due to interference at receiver \( C \), in which case \( w \) and \( v' \) also interfere with each other. \( \blacksquare \)
Let $G, H$ be simple, undirected graphs, and let $v \in V(G)$. Let $G^v \leftarrow H$ be the graph obtained by taking the disjoint union of $G - v$ and $H$, and joining each vertex in $H$ to each vertex in $G_1(v)$, where $G_1(v)$ denotes the neighbors in $G$ of vertex $v$. We say $G^v \leftarrow H$ is the graph obtained from $G$ by replacing vertex $v$ by $H$.

**Lemma 13.** Let $G$ be a simple, undirected graph, and let $G^v \leftarrow K_r$ denote the graph obtained from $G$ by replacing vertex $v$ of $G$ by the complete graph $K_r$ ($r \geq 1$). Then $\sigma(G^v \leftarrow K_r) = \sigma(G)$.

**Proof:** Recall that $\sigma(G)$ is defined as the number of leaf vertices in a maximum sized induced star in $G$. If $v$ is the center vertex of a maximum sized induced star of $G$, then replacing $v$ by $K_r$ will replace the star with $r$ stars of the same size. Suppose $v$ is a leaf vertex of a maximum sized induced star of $G$. Since at most one vertex from a clique can belong to an independent set, and because the neighbors of $K_r$ in $G^v \leftarrow K_r$ are the same as the neighbors of $v$ in $G$, we have $\sigma(G^v \leftarrow K_r) = \sigma(G)$.

**Remark 14.** Lemma 12 says that in the conflict graph of a line network, vertices $v = (A, \{B, C\})$ and $w = (D, E)$ interfere with each other if and only if vertices $v' = (A, C)$ and $w = (D, E)$ interfere with each other. A similar proof can be given to show that vertices $v = (A, \{B, C\})$ and $w = (D, \{E, F\})$ interfere with each other if and only if vertices $v' = (A, C)$ and $w' = (D, F)$ interfere with each other. Thus, when constructing the conflict graph of a line network, it can be assumed that the given set $\{(i_k, j_k) : k \in K\}$ of valid transmissions is such that each transmission is a unicast transmission, i.e. that $|J_k| = 1$, for all $k \in K$. If the original set of valid transmissions contained both a multicast transmission $(A, \{B, C\})$ and the corresponding unicast transmission $(A, C)$, then the new set of valid transmissions would contain two copies of the unicast transmission $(A, C)$. Lemma 12 implies that the conflict graph constructed for this new set of (unicast) transmissions is isomorphic to the conflict graph constructed for the original set. Furthermore, by Lemma 13 replacing a single vertex in a graph with a complete graph on two vertices preserves the induced star number of the graph. If there are two vertices in the conflict graph corresponding to the same unicast transmission, one of these vertices can be removed without affecting the induced star number. Hence, as far as results (or bounds) on the induced star number of the conflict graph of a line network are concerned, it can be assumed without loss of generality that the given set of valid transmissions consists only of distinct unicast transmissions.

**Theorem 15.** Consider a line network whose nodes are $A_1, A_2, \ldots, A_n$, in order from left to right. Suppose each node can transmit to at most two nodes to its right and all transmissions are in the direction from $A_1$ to $A_n$. Then, under the protocol interference model (Construction 9), the induced star number of the conflict graph of this line network is at most 3. Further, this bound is best possible.

**Proof:** Without loss of generality, assume the transmission radius is 1. By way of contradiction, suppose the star $K_{1,4}$ is an induced subgraph of the conflict graph $G_c$, with center vertex $v_1$ and leaf vertices $v_2, v_3, v_4$ and $v_5$. By Lemma 12 and Remark 14
it can be assumed without loss of generality that the given set of valid transmissions consists of distinct unicast transmissions. Hence, we may assume that the vertices $v_i$ correspond to distinct unicast transmissions.

Suppose each transmission $v_i$ is a unicast transmission from a node $A_i$ at position $a_i$ to a node $B_i$ at position $b_i$. Recall from the proof of Lemma 10 that if transmissions $v_i$ and $v_j$ are nonadjacent in $G_c$, then the closed intervals $[a_i, b_i]$ and $[a_j, b_j]$ are disjoint. Since $\{v_2, v_3, v_4, v_5\}$ is an independent set in $G_c$, the closed intervals $[a_i, b_i]$, for $i = 2, 3, 4, 5$, are disjoint. Without loss of generality, assume the closed intervals are in order $v_2, v_3, v_4, v_5$ from left to right, as shown in Figure 4.

Now consider a few cases, depending on the location $a_1$ of the transmitter $A_1$ of the transmission $v_1 = (A_1, B_1)$. Let $s_i := b_i - a_i$ denote the distance between the transmitter and receiver for transmission $v_i$, $i = 1, \ldots, 5$. Since $v_2v_3 \notin E(G_c)$, by Lemma 10 one obtains $a_3 - b_2 > s_2$. We claim $a_1 < a_3$. If $a_1 \geq a_3$, then $a_1 - b_2 \geq a_3 - b_2 > s_2$, whence $v_1v_2 \notin E(G_c)$ by Lemma 10, a contradiction. Thus, $a_1 < a_3$. Also, if $b_1 > b_3$, then the four nodes $A_1, A_3, B_3, B_1$ are positioned in that order from left to right, which implies that there exists a node, namely $A_1$, which is able to communicate with up to 3 nodes to its right, a contradiction. Hence, $b_1 \leq b_3$. But then $a_5 - b_3 > 1$ and $b_1 \leq b_3$ imply $a_5 - b_1 > 1$, whence $v_1v_5 \notin E(G_c)$, a contradiction.

It has been proved that the conflict graph $G_c$ does not contain a $K_{1,4}$ as an induced subgraph. Hence, $\sigma(G_c) \leq 3$. This bound is best possible because, by Proposition 11 there exists a line network for which the conflict graph $G_c$ contains a claw.

5. Concluding remarks

Sufficient conditions for distributed admission control were obtained, and it was shown that under the primary interference model, there exists a distance-$d$ distributed algorithm whose worst-case performance is a factor of $(2d + 3)/(2d + 2)$ away from optimal. This bound is independent of the structure of the network graph.

It was shown that for line networks, under the protocol interference model, the row constraints are a factor of at most 3 away from optimal and that this bound is best possible. A line network was given for which the conflict graph contains a claw. This implies that the polynomial time scheduling algorithms in the literature devised for claw-free graphs are not directly applicable to solving the scheduling problem.
These results can be extended in several directions. First, in the distance-$d$ distributed proposed in this work, it was assumed that wireless interference was modeled as primary interference. This can be extended to conflict graphs constructed from other interference models. Second, these results can be extended to more general interference models such as hypergraphs. For instance, when the interference is such that any two of some three wireless links can be simultaneously active, a hyperedge consisting of the three links captures the minimal forbidden set of links. Third, the single-hop case was considered in the present work, and one can extend these results to the setting of multi-hop networks. Fourth, the results on line networks under the protocol interference model can be extended to other topologies. Finally, much work has been done on computing maximum weight independent sets in claw-free graphs and its special case of line graphs; the induced star number of these graphs is at most 2. Designing efficient, distributed scheduling algorithms for graphs having bounded induced star number is an open direction.

References


