Fault tolerant supergraphs with automorphisms

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Abstract

Given a basic graph $Y$ and a desired level of fault-tolerance $k$, an objective in fault-tolerant system design is to construct a supergraph $X$ such that the removal of any $k$ nodes from $X$ leaves a graph containing $Y$. In order to reconfigure around faults when they occur, it is also required that any two subsets of $k$ nodes of $X$ are in the same orbit of the action of its automorphism group. In this paper, we prove that such a supergraph must be the complete graph. The proof uses a result due to Cameron on $k$-homogeneous groups. Our work resolves an open problem in the literature.

Index terms — fault-tolerant system design; automorphisms; automorphic reconfiguration; graph theory; $k$-homogeneous groups.

1. Introduction

The interconnection network of a computing system is modeled as a graph $X = (V, E)$ whose vertices correspond to processors and with two vertices being adjacent whenever the corresponding two processors are connected by a direct communication link (cf. [1], [2]). In order to execute an algorithm on this computing system, it is required that the architecture $X$ contain a given basic graph $Y$ as a subgraph. If some of the nodes of $X$ become faulty, in order to continue operation it is required that the functioning part of the network still contain the basic graph $Y$. We assume the basic graph $Y$ is nonempty, i.e. it contains at least one edge.

Let $Y$ be a nonempty graph on $n$ vertices. A graph $X$ is said to be a $k$-fault-tolerant realization of $Y$ if $X$ can be obtained from $Y$ by adding a set of $k$ new vertices (called spare nodes) and some edges so that the resulting graph $X$ has the property that the removal of any $k$ vertices from $X$ leaves a graph which still contains $Y$ (cf. [3]). In other words, $X$ is a $k$-fault-tolerant realization of $Y$ if $X$ has $n + k$ vertices and $X - W$ contains a subgraph isomorphic to $Y$ for each $k$-subset $W \subseteq V(X)$. In this case, if any $k$ nodes of $X$ become faulty, the network corresponding to the nonfaulty nodes of $X$ contains the architecture $Y$ and hence can continue to operate. In this sense, the architecture $X$ can tolerate up to $k$ node failures.

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In order to achieve so-called automorphic reconfiguration (cf. [3] p.253), it is also
required that, when \( k \) or fewer faults occur, there exists an automorphism of the
graph \( X \) that maps the spare nodes to the faulty nodes. In graph-theoretic terms, the
graph \( X \) must satisfy the property that if \( A \) and \( B \) are any two \( k \)-subsets of \( V(X) \),
then there is an automorphism of \( X \) that maps \( A \) to \( B \).

Automorphic reconfiguration can be described using the terminology of group
actions, as follows. Let \( G \) be a group acting on a set \( \Omega \) and let \( k \geq 2 \). Then, \( G \) also acts
naturally on the set \( \Omega^{(k)} \) of all \( k \)-subsets of \( \Omega \) by the rule \( \{\alpha_1, \ldots, \alpha_k\}^g = \{\alpha_1^g, \ldots, \alpha_k^g\} \)
for each \( g \in G \) and each \( \{\alpha_1, \ldots, \alpha_k\} \in \Omega^{(k)} \). A group \( G \) acting on a set \( \Omega \) is said
to be \( k \)-homogeneous if given any two \( k \)-subsets \( A, B \subseteq \Omega \), there exists an element
\( g \in G \) such that \( A^g = B \). In other words, a group \( G \) acting on a set \( \Omega \) is said to be \( k \)-
homogeneous if \( G \) acts on \( \Omega^{(k)} \) transitively. Observe that the property that the graph
\( X \) in the previous paragraph must satisfy is that its automorphism group \( \text{Aut}(X) \) be
\( k \)-homogeneous. What we call \( k \)-homogeneous here is referred to as \( k \)-subtransitive
in [3]. For an introduction to group actions, we refer the reader to [2, Chapter 1]; for
an introduction to \( k \)-homogeneous groups, see [2, Sections 2.1 and 9.4].

Thus, given a basic graph \( Y \) and a desired level of fault-tolerance \( k \), our objective
is to construct a graph \( X \) such that \( X \) is a \( k \)-fault-tolerant realization of \( Y \) and such
that \( \text{Aut}(X) \) is \( k \)-homogeneous. Dutt and Hayes settled this problem for the case
\( k = 2 \) by proving the following result:

**Theorem 1.** [3, Theorem 2] If \( Y \) is a nonempty graph, \( X \) is a 2-fault-tolerant realization
of \( Y \) and \( \text{Aut}(X) \) is 2-homogeneous, then \( X \) is the complete graph.

Dutt and Hayes posed the problem of generalizing the \( k = 2 \) result of Theorem 1 to
arbitrary \( k \) (cf. [3] p. 253)). In this paper, we resolve this problem (cf. Theorem 3 below). Our proof uses the following result due to Cameron:

**Theorem 2.** [4, Theorem 2.2] [2, Theorem 9.4A] Let \( G \) be a permutation group acting
on a set \( \Omega \). Let \( m, k \) be integers with \( 0 \leq m \leq k \) and \( m + k \leq |\Omega| \). Then, \( G \) has at
least as many orbits in \( \Omega^{(k)} \) as it has in \( \Omega^{(m)} \).

The following is the main result of this paper:

**Theorem 3.** Let \( k \geq 2 \). If \( Y \) is a nonempty graph, \( X \) is a \( k \)-fault-tolerant realization
of \( Y \) and \( \text{Aut}(X) \) is \( k \)-homogeneous, then \( X \) is the complete graph.

**Proof:** The case \( k = 2 \) is addressed in Theorem 1 so assume \( k \geq 3 \). Let \( Y \) be a graph
on \( n \) vertices. Here, \( n \geq 2 \) since \( Y \) contains at least one edge. Note that \( X \) is a graph
on \( n + k \) vertices. Let \( G = \text{Aut}(X) \) and let \( \Omega = V(X) \). By hypothesis, the action
of \( G \) on the set \( \Omega \) is \( k \)-homogeneous. Thus, the number of orbits of \( G \) on \( \Omega^{(k)} \) is 1.
Since \( 2 \leq n \), \( 2 + k \leq n + k = |\Omega| \). Also, \( 2 \leq k \). Hence, by Theorem 2, the number of
orbits of \( G \) on \( \Omega^{(2)} \) is also 1. Equivalently, the action of \( G \) on \( \Omega \) is \( 2 \)-homogeneous.

Let \( \{u, v\} \) be an edge in \( Y \). Then \( \{u, v\} \) is an edge in \( X \). Let \( a \) and \( b \) be distinct
vertices of \( X \). Because \( G \) is \( 2 \)-homogeneous, there is an element \( g \in G \) that maps
\( \{u, v\} \) to \( \{a, b\} \). The automorphism \( g \) preserves adjacency, whence \( \{a, b\} \) is an edge
of $X$. This proves that any two distinct vertices of $X$ are adjacent, i.e. $X$ is the complete graph.

Theorem 1 and Theorem 3 imply that it is very expensive to have an interconnection network which is $k$-fault-tolerant and which also supports automorphic reconfiguration because such an interconnection network must be the complete graph.

References


